Quantitative Methods in Geography

Geo 340

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Probability

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Sample space

To help think about probabilities, we can use the notion of a sample space. A sample space is the set of all elementary outcomes of a probabilistic experiment. For example, consider the set of all possible outcomes of tossing a single fair coin twice in succession:

$$\mathcal{S} = \{HH, HT, TH, TT\}$$

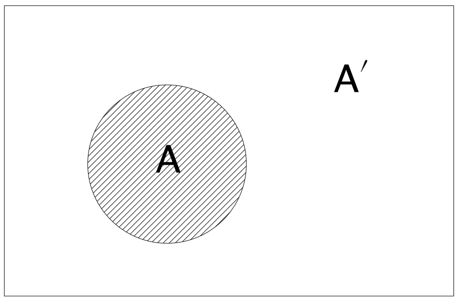
Event and event space

An event is a subset of the sample space. For example, an event within the sample space of two tosses of a coin might be getting at least one tail. An event space is the subspace of sample space that contains all elementary outcomes comprising the event.

$$A = \{HT, TH, TT\}$$

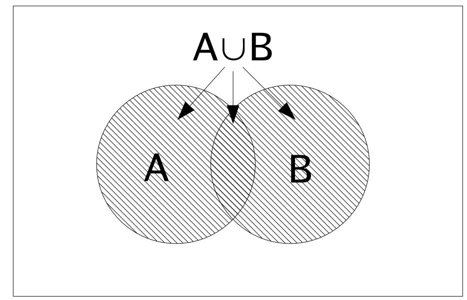
Complement

In this Venn diagram, the circle encloses all of the elementary outcomes that comprise event A. The elementary outcomes that are not part of event A are outside of the circle and comprise the complement of A which is called A'.



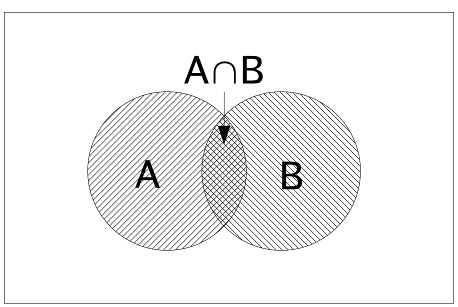
Union

The union of two events, $A \cup B$, is the set of all elementary outcomes that belong to at least one of the events A or B. This is equivalent to a logical, non-exclusive "or" operation.



Intersection

The set of elementary outcomes that are common to two events are called the intersection of those events, $A \cap B$. This is equivalent to a logical "and" operation. Mutually exclusive events are those where $A \cap B = \varnothing$.



Probability rules

$$0 \leq P(E_i) \leq 1$$

for $i = 1, 2, 3 \ldots n$ and E_i is an elementary outcome in sample space $\mathcal{S}$.

$$P(A) = \sum_{E_i \in A} P(E_i)$$

for any event A.

$$P(\mathcal{S}) = 1$$

$$P(\varnothing) = 0$$

Rules (cont'd)

From these basic rules, it follows that:

$$\sum_{i=1}^{n} P(E_i) = 1$$

$$0 \leq P(A) \leq 1$$

And if, and only if, $A \cap B = \varnothing$, then $P(A \cap B) = 0$.

Counting rules: product rule

To determine the number of equally likely elementary outcomes when selecting one object from each of k groups, each containing n_j distinct objects we find the product of the n_j's:

$$\prod_{j=1}^{k} n_j$$

For example, if $k = 3$ and $n_1 = 10, n_2 = 7, n_3 = 4$, there are $10 \cdot 7 \cdot 4 = 280$ different ways that we could select one object from each group.

Computing probabilities

$$P(A) = \frac{m}{n}$$

where m is the number of equally likely elementary outcomes in event A and n is the number of equally likely elementary outcomes in the sample space.

To find the numbers m and n, we need some counting rules.

Counting rules: combinations

When selecting a subset of k objects from a set of n distinct objects *without* replacement where $k \leq n$ and order is *not* important, we count the possibilities using the combinations rule:

$$C(n,k) = \binom{n}{k} = C_k^n = \frac{n!}{k!(n-k)!}$$

For example, if we want to select three objects from a set of six distinct objects without replacement and order not important, we calculate:

$$C_3^6 = \frac{6!}{3!(6-3)!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = 20$$

Combinations (cont'd)

For another example that is small enough to do by hand, let's find out how many ways we can select two objects from a set of four distinct objects, without replacement and order not important. If the objects are labeled *a*, *b*, *c* and *d*, these are the possible outcomes:

ab		
ac	*bc*	
ad	*bd*	*cd*

We can check our count of six possibilities by applying the formula:

$$C_2^4 = \frac{4!}{2!(4-2)!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 2 \cdot 1} = 6$$

Counting rules: permutations

When selecting a subset of k objects from a set of n distinct objects *without* replacement where $k \leq n$ and order *is* important, we count the possibilities using the permutations rule:

$$P_k^n = \frac{n!}{(n-k)!}$$

For example, if we want to select three objects from a set of six distinct objects without replacement, but order *is* important, we calculate:

$$P_3^6 = \frac{6!}{(6-3)!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 120$$

Permutations (cont'd)

We can redo our example of selecting two objects from a set of four distinct objects without replacement, but this time order is important.

	ba	*ca*	*da*
ab		*cb*	*db*
ac	*bc*		*dc*
ad	*bd*	*cd*	

We can check our count of twelve possibilities by applying the formula:

$$P_2^4 = \frac{4!}{(4-2)!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = 12$$

Selecting with replacement

So far we have only discussed selecting *without* replacement. If we need to count the possibilities when selecting a subset of k objects from a set of n distinct objects *with* replacement and order important, we simply calculate n^k.

For example, if the order of our selections is important and if we want to select three objects from a set of six distinct objects and replace each object after selecting it so that it has an equal chance of being selected again, then there are $6^3 = 216$ possibilities.

An example with replacement, order important

Once again we redo our example of selecting two objects from a set of four distinct objects. Now we do it with replacement and order is important.

aa	*ba*	*ca*	*da*
ab	*bb*	*cb*	*db*
ac	*bc*	*cc*	*dc*
ad	*bd*	*cd*	*dd*

We can check our count of sixteen possibilities by applying the formula:

$$n^k = 4^2 = 16$$

Selecting with replacement, order not important

We can count the number of possibilities of selecting a subset of k objects from a set of n distinct objects with replacement and order *not* important by calculating:

$$C_k^{n-1+k} = \frac{(n-1+k)!}{k!(n-1)!}$$

This gives the number of possibilities, but they are *not* all equally likely and, therefore, this type of selection is not often used.

Example with replacement, order not important

For the last time, we redo our example of selecting two objects from a set of four distinct objects. Now we do it with replacement, but order is not important.

aa			
ab	*bb*		
ac	*bc*	*cc*	
ad	*bd*	*cd*	*dd*

We can check our count of ten possibilities with the formula:

$$C_k^{n-1+k} = C_2^{4-1+2} = \frac{5!}{2!(4-1)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = 10$$

But they are not all equally likely. There are two ways to draw each element except *aa*, *bb*, *cc*, and *dd*, which can each only be selected one way.

Summary of rules for selecting from one set

	Replacement	No replacement
Order important	n^k	P_k^n
Order not important	C_k^{n-1+k}	C_k^n

Hypergeometric rule

This is a combination of the product rule and the combinations rule. To count the number of elementary outcomes when selecting r_j objects from each of k groups each containing n_j distinct objects, we use the combinations rule to find the number of elementary outcomes in each of the k groups, $C^{n_j}_{r_j}$, and then find the product of the k combinations:

$$\prod_{j=1}^{k} C^{n_j}_{r_j} = C^{n_1}_{r_1} \cdot C^{n_2}_{r_2} \cdot C^{n_3}_{r_3} \cdots C^{n_k}_{r_k}$$

Statistical independence

Conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Two events, A and B defined over sample space $\mathcal{S}$, are statistically independent if and only if $P(A|B) = P(A)$ or $P(B|A) = P(B)$.

Basic probability theorems

Addition theorem:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Complementation theorem:

$$P(A) = 1 - P(A')$$

or

$$P(A') = 1 - P(A)$$

Multiplication theorem

The multiplication theorem comes from the conditional probability rule:

$$P(A \cap B) = P(A|B) \cdot P(B)$$

$$P(A \cap B) = P(B|A) \cdot P(A)$$

For independent events, $P(A|B) = P(A)$, so

$$P(A \cap B) = P(A) \cdot P(B)$$